

# A Trotter-Kato Theorem for Quantum Markov Limits

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## Abstract

Using the Trotter-Kato theorem we prove the convergence of the unitary dynamics generated by an increasingly singular Hamiltonian in the case of a single field coupling. The limit dynamics is a quantum stochastic evolution of Hudson-Parthasarathy type, and we establish in the process a graph limit convergence of the pre-limit Hamiltonian operators to the Chebotarev-Gregoratti-von Waldenfels Hamiltonian generating the quantum Itô evolution.

## 1 Introduction

In the situation of regular perturbation theory, we typically have a Hamiltonian interaction of the form  $H = H_0 + H_{\text{int}}$  with associated strongly continuous one-parameter unitary groups  $U_0(t) = e^{-itH_0}$  (the free evolution) and  $U(t) = e^{-itH}$  (the perturbed evolution), then we transform to the Dirac interaction picture by means of the unitary family  $V(t) = U_0(-t)U(t)$ . Although  $V(\cdot)$  is strongly continuous, it does not form a one-parameter group but instead yields what is known as a left  $U_0$ -cocycle:

$$V(t+s) = U_0(s)^\dagger V(t) U_0(s) V(s). \quad (1)$$

One obtains the interaction picture dynamical equation

$$i \frac{d}{dt} V(t) = \Upsilon(t) V(t), \quad (2)$$

where  $\Upsilon(t) = U_0(t)^\dagger H_{\text{int}} U_0(t)$ .

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More generally, we may have a pair of unitary groups  $U(\cdot)$  and  $U_0(\cdot)$  with Stone generators  $H$  and  $H_0$  respectively, but where the intersection of the domains of the generators are not dense. This is the situation of a singular perturbation. In this case we cannot expect the Dirac picture dynamical equation (2) to be anything but formal since the difference  $H_{\text{int}} = H - H_0$  is not densely defined.

Remarkably, the steps above can be reversed even for the situation of singular perturbations. If we assume at the outset a fixed free dynamics  $U_0(\cdot)$ , with Stone generator  $H_0$ , and a strongly continuous unitary left  $U_0$ -cocycle  $V(\cdot)$ , then  $U(t) = U_0(t)V(t)$  will then form a strongly continuous one-parameter unitary group with Stone generator  $H$ . In practice however the problem of reconstructing  $H$  from the prescribed  $H_0$  and  $V(\cdot)$  will be difficult.

In the situation of quantum stochastic evolutions introduced by Hudson and Parthasarathy [6], we have a strongly continuous adapted process  $V(\cdot)$  satisfying a quantum stochastic differential equation (including Wiener and Poisson noise as special commutative cases) in place of (2), and the solution constitutes a cocycle with respect to the time-shift maps  $U_0 \equiv \Theta$  (see below). Nevertheless,  $V(\cdot)$  arises as the Dirac picture evolution for a singular perturbation of a unitary  $U(\cdot)$  with some generator  $H$  with respect to the time-shift: it was a long standing problem to find an explicit form for  $H$  which was finally resolved by Gregoratti [4], see also [5].

The purpose of this paper is to approximate the singular perturbation arising in quantum stochastic evolution models by a sequence of regular perturbation models. That is, to construct a sequence of Hamiltonians  $H^{(k)} = H_0 + H_{\text{int}}^{(k)}$  yielding a regular perturbation  $V^{(k)}(\cdot)$  converging to a singular perturbation  $V(\cdot)$  in some controlled way. We exploit the fact that the limit Hamiltonian is now known through the work of Chebotarev [1] and Gregoratti [4]. The strategy is to employ the Trotter-Kato theorem which guarantees strong uniform convergence of the unitaries once graph convergence of the Hamiltonians is established.

## 1.1 Quantum Stochastic Evolutions

The seminal work of Hudson and Parthasarathy [6] on quantum stochastic evolutions lead to explicit constructions of unitary adapted quantum stochastic processes  $V$  describing the open dynamical evolution of a system with a singular Boson field environment. We fix the system Hilbert space  $\mathfrak{h}$  and model the environment as having  $n$  channels so that the underlying Fock space is  $\mathfrak{F} = \Gamma(\mathbb{C}^n \otimes L^2(\mathbb{R}))$ . Here  $\Gamma(\mathfrak{H})$  denotes the symmetric (boson) Fock space over a one-particle space  $\mathfrak{H}$ : we set the inner product as  $\langle \Psi | \Phi \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \langle \Psi_m | \Phi_m \rangle$  and take the exponential vectors to be defined as  $(\otimes_s$  denoting a symmetric tensor product)

$$e(f) = (1, f, f \otimes_s f, f \otimes_s f \otimes_s f, \dots)$$

with test function  $f \in \mathfrak{H}$ . Here the one particle space is  $L^2(\mathbb{R})$ , the space of complex-valued square-integrable functions on  $\mathbb{R}$ . We define the operators

$$\begin{aligned}\Lambda^{00}(t) &\triangleq t, \\ \Lambda^{10}(t) &= A^\dagger(t) \triangleq a^\dagger(1_{[0,t]}), \\ \Lambda^{01}(t) &= A(t) \triangleq a(1_{[0,t]}), \\ \Lambda^{11}(t) &= \Lambda(t) \triangleq d\Gamma(\chi_{[0,t]}),\end{aligned}$$

where  $1_{[0,t]}$  is the characteristic function of the interval  $[0, t]$  and  $\chi_{[0,t]}$  is the operator on  $L^2(\mathbb{R})$  corresponding to multiplication by  $1_{[0,t]}$ . Hudson and Parthasarathy [6] have developed a quantum Itô calculus where the basic objects are integrals of adapted processes with respect to the fundamental processes  $\Lambda^{\alpha\beta}$ . The quantum Itô table is then

$$d\Lambda^{\alpha\beta}(t) d\Lambda^{\mu\nu}(t) = \hat{\delta}_{\beta\mu} d\Lambda^{\alpha\nu}(t)$$

where  $\hat{\delta}_{\alpha\beta}$  is the Evans-Hudson delta defined to equal unity if  $\alpha = \beta = 1$  and zero otherwise. This may be written as

$\times$	$dA$	$d\Lambda$	$dA^\dagger$	$dt$
$dA$	0	$dA$	$dt$	0
$d\Lambda$	0	$d\Lambda$	$dA$	0
$dA^\dagger$	0	0	0	0
$dt$	0	0	0	0

In particular, we have the following theorem [6].

**Theorem 1** *There exists a unique solution  $V(\cdot, \cdot)$  to the quantum stochastic differential equation*

$$V(t, s) = I + \int_s^t dG(\tau) V(\tau, s) \quad (3)$$

( $t \geq s \geq 0$ ) where

$$dG(t) = G_{\alpha\beta} \otimes d\Lambda^{\alpha\beta}(t)$$

with  $G_{\alpha\beta} \in \mathfrak{B}(\mathfrak{h})$ . (We adopt the convention that we sum repeated Greek indices over the range 0, 1.)

In particular, set  $V(t) = V(t, 0)$  then we have the quantum stochastic differential equation  $dV(t) = dG(t) V(t)$  which replaces the regular Dirac picture dynamical equation (2).

We refer to  $\mathbf{G} = [G_{\alpha\beta}] \in \mathfrak{B}(\mathfrak{h} \oplus \mathfrak{h})$ , as the *coefficient matrix*, and  $V$  as the left process generated by  $\mathbf{G}$ . The conditions for the process  $V$  to be unitary are that  $\mathbf{G}$  takes the form, with respect to the decomposition  $\mathfrak{h} \oplus \mathfrak{h}$ ,

$$\mathbf{G} = \begin{bmatrix} -\frac{1}{2}\mathbf{L}^\dagger\mathbf{L} - i\mathbf{H} & -\mathbf{L}^\dagger\mathbf{S} \\ \mathbf{L} & \mathbf{S} - I \end{bmatrix} \quad (4)$$

where  $S \in \mathfrak{B}(\mathfrak{h})$  is a unitary,  $L \in \mathfrak{B}(\mathfrak{h})$  and  $H \in \mathfrak{B}(\mathfrak{h})$  is self-adjoint. We may write in more familiar notation [6]

$$dG(t) = \left( -\frac{1}{2}L^\dagger L - iH \right) \otimes dt - L^\dagger S \otimes dA(t) + L \otimes dA^\dagger(t) + (S - I) \otimes d\Lambda(t).$$

We denote the shift map on  $L^2(\mathbb{R})$  by  $\theta_t$ , that is  $(\theta_t)f(\cdot) = f(\cdot + t)$  and its second quantization as  $\Theta_t = I \otimes \Gamma(\theta_t)$ . It then turns out that  $\Theta_\tau^\dagger V(t, s) \Theta_\tau = V(t + \tau, s + \tau)$  and so  $V(t) = V(0, t)$  is a left unitary  $\Theta$ -cocycle and that there must exist a self-adjoint operator  $H$  such that

$$\Theta_t V(t) \equiv e^{-iHt}$$

for  $t \geq 0$ . (For  $t < 0$  one has  $V(-t)^\dagger \Theta_{-t} \equiv e^{-iHt}$ .) Here  $H$  will be a singular perturbation of generator of the shift, and its characterization was given by Gregoratti [4]. See also [14].

## 1.2 Physical Motivation

As a precursor to and motivation for further approximations, we fix on a simple model of a quantum mechanical system coupled to a boson field reservoir  $R$ . In the Markov approximation we assume that the auto-correlation time of the field processes vanishes in the limit: this includes weak coupling (van Hove) and low density limits. The Hilbert space for the field is the Fock space  $\mathcal{F}_R = \Gamma(\mathcal{H}_R^1)$  with one-particle space  $\mathcal{H}_R^1 = L^2(\mathbb{R})$  taken as the momentum space. (For convenience we consider a one-dimensional situation because this is the setting studied in this paper but of course  $\mathbb{R}^3$  is particularly relevant physically.) It is convenient to write annihilation operators formally as  $A_R(g) = \int_{\mathbb{R}} g(p)^* a_p dp$  where  $[a_p, a_{p'}^\dagger] = \delta(p - p')$ .

In particular, let us fix a function  $g \in L^2(\mathbb{R})$ , and set

$$a(t, k) = \sqrt{k} \int e^{-i\omega(p)tk} g(p)^* a_p dp$$

where  $\omega = \omega(p)$  is a given function (determining the dispersion relation for the free quanta) and  $k$  is a dimensionless parameter rescaling time. We have the commutation relations

$$[a(s, k), a(t, k)^\dagger] = k \rho(k(t - s))$$

where

$$\rho(\tau) \equiv \int |g(p)|^2 e^{i\omega(p)\tau} dp.$$

The limit  $k \rightarrow \infty$  leads to singular commutation relations, and it is convenient to introduce smeared fields

$$A(\varphi, k) = \int \varphi(t)^* a(t, k) dt$$

in which case we have the two-point function (and define an operator  $C_k$  by)

$$\left[ A(\varphi, k), A(\psi, k)^\dagger \right] = \int dt dt' \varphi(t)^* k \rho(k(t-t')) \psi(t') \equiv \langle \varphi | C_k \psi \rangle$$

For  $\rho$  integrable, we expect

$$\lim_{k \rightarrow \infty} \left[ A(\varphi, k), A(\psi, k)^\dagger \right] = \gamma \int dt \varphi(t)^* \psi(t)$$

where  $\gamma = \int_{-\infty}^{\infty} \rho(\tau) d\tau = 2\pi \int |g|^2(p) \delta(\omega(p)) dp \geq 0$ . When  $\gamma = 1$ , the  $A(\varphi, k)$  are smeared versions of the annihilators on  $\Gamma(L^2(\mathbb{R}))$ .

The limit  $k \uparrow \infty$  corresponds to the smeared field becoming singular and this leads to a quantum Markovian approximation. The formulation of such models was first given and treated in a systematic way by Accardi, Frigero and Lu who developed a set of powerful quantum functional central limit theorems including the weak coupling [7] and low density [8] regimes. Theorem 2 is an extension of these which includes both quantum diffusion and jump terms [2, 3].

**Theorem 2** *Let  $(E_{\alpha\beta})$  be bounded operators on a fixed separable Hilbert space  $\mathfrak{h}$  labeled by  $\alpha, \beta \in \{0, 1\}$  with  $E_{\alpha\beta}^\dagger = E_{\beta\alpha}$  and  $\|E_{11}\| < 2$ . Let*

$$\Upsilon(t, k) = E_{11} \otimes a(t, k)^\dagger a(t, k) + E_{10} \otimes a(t, k)^\dagger + E_{01} \otimes a(t, k) + E_{00} \otimes I$$

and

$$e(\varphi, k) = \exp \left\{ A(\varphi, k) - A(\varphi, k)^\dagger \right\} \Omega_R$$

with  $\Omega_R$  the Fock vacuum of  $\mathcal{F}_R$ . The solution  $V(t, k)$  to the equation

$$\frac{d}{dt} V(t, k) = -i \Upsilon(t, k) V(t, k), \quad V(0, k) = I,$$

exists and we have the limit

$$\lim_{k \rightarrow \infty} \langle u_1 \otimes e(\varphi, k) | V(t, k) | u_2 \otimes e(\psi, k) \rangle = \langle u_1 \otimes e(\varphi) | V(t) | u_2 \otimes e(\psi) \rangle$$

for all  $u_1, u_2 \in \mathfrak{h}$  and  $\varphi, \psi \in L^2(\mathbb{R})$ , where  $V$  is a unitary adapted process on  $\mathfrak{h} \otimes \Gamma(\mathbb{C}^n \otimes L^2(\mathbb{R}))$  with coefficient matrix  $\mathbf{G}$  given by

$$\mathbf{G} = -i\mathbf{E} - i\frac{1}{2}\mathbf{G} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{E} \quad (5)$$

where we assume  $\int_{-\infty}^0 \rho(\tau) d\tau = \frac{1}{2}$ .

The proof of the Theorem is given in [2] and requires a development and a uniform estimation of the Dyson series expansion. Summability of the series requires that  $\|E_{11}\| < 2$ .

The triple  $(S, L, H)$  from (4) obtained through (5) is

$$\begin{aligned} S &= \frac{I - \frac{i}{2}E_{11}}{I + \frac{i}{2}E_{11}}, & L &= -\frac{i}{I + \frac{i}{2}E_{11}}E_{10}, \\ H &= E_{00} + E_{01}\text{Im}\left\{\frac{1}{I + \frac{i}{2}E_{11}}\right\}E_{10}. \end{aligned} \quad (6)$$

Our objective is reappraise Theorem 2, where we will prove a related result by an alternative technique. Using the Trotter-Kato theorem, we will establish a stronger mode of convergence (uniformly on compact intervals of time and strongly in the Hilbert space) by means of a graph convergence of the Hamiltonians. The new approach has the advantage of been simpler and is likely to be more readily extended to other cases, for instance a continuum of input channels as originally treated in [6], which cannot be treated by the perturbative techniques used in the proof of Theorem 2.

## 2 Trotter-Kato Theorems for Quantum Stochastic Limits

Our main results will employ the Trotter-Kato theorem, which we recall next in a particularly convenient form. See [9], Theorem 3.17, or [12], Chapter VIII.7.

**Theorem 3 (Trotter-Kato)** *Let  $\mathcal{H}$  be a Hilbert space and let  $U^{(k)}(\cdot)$  and  $U(\cdot)$  be strongly continuous one-parameter groups of unitaries on  $\mathcal{H}$  with Stone generators  $H^{(k)}$  and  $H$ , respectively. Let  $\mathcal{D}$  be a core for  $H$ . The following are equivalent*

1. *For all  $f \in \mathcal{D}$  there exist  $f^{(k)} \in \text{Dom}(H^{(k)})$  such that*

$$\lim_{k \rightarrow \infty} f^{(k)} = f, \quad \lim_{k \rightarrow \infty} H^{(k)} f^{(k)} = Hf.$$

2. *For all  $0 \leq T < \infty$  and all  $f \in \mathcal{H}$  we have*

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \left\| \left( U^{(k)}(t) - U(t) \right) f \right\| = 0.$$

The theorem yields a strong uniform convergence if we can establish graph convergence of the Hamiltonians. We now present the Trotter-Kato theorems for the class of problems that interest us, treating the first and second quantized problems in sequence.

### 2.1 First Quantization Example

**Definition 4** *Let  $g \in C_c^\infty(\mathbb{R})$ , i.e., an infinitely differentiable function with compact support, such that  $\int_{-\infty}^{\infty} g(s)ds = 1$ . We define  $\rho(t) = \int_{\mathbb{R}} g(s)^* g(s+t)ds$ .*

Moreover, for all  $k > 0$ , we define functions  $g^{(k)}$  and  $\rho^{(k)}$  by

$$g^{(k)}(t) = k g(kt), \quad \rho^{(k)}(t) = k \rho(kt), \quad t \in \mathbb{R}.$$

Furthermore, we define two complex numbers by  $\kappa_+ := \int_0^\infty \rho(s) ds$  and  $\kappa_- := \int_{-\infty}^0 \rho(s) ds$ .

Note that  $\kappa_+ + \kappa_- = 1$  and that  $\kappa_+$  and  $\kappa_-$  are complex conjugate:  $\kappa_+ = (\kappa_-)^*$  (substitute  $-s$  for  $s$ ), hence  $\kappa_\pm = \frac{1}{2} \pm i\sigma$  with  $\sigma$  real. The choice of  $\rho$  is such that  $\langle g|g * f \rangle = \langle \rho|f \rangle$ , where  $(g * f)(t) = \int_{-\infty}^\infty g(s)f(t-s)ds$  is the usual convolution.

Let  $\mathfrak{h}$  be a Hilbert space and let  $E$  be a bounded self-adjoint operator on  $\mathfrak{h}$ . We consider the following family of operators on  $L^2(\mathbb{R}; \mathfrak{h}) \simeq \mathfrak{h} \otimes L^2(\mathbb{R})$ :

$$\begin{aligned} H^{(k)} &= i \partial + E |g^{(k)}\rangle \langle g^{(k)}| \simeq I \otimes i \partial + E \otimes |g^{(k)}\rangle \langle g^{(k)}|, \\ \text{Dom}(H^{(k)}) &= W^{1,2}(\mathbb{R}; \mathfrak{h}), \end{aligned} \quad (7)$$

where  $W^{1,2}(X; \mathfrak{h})$ ,  $X \subseteq \mathbb{R}$ , denotes the Sobolev space of  $\mathfrak{h}$ -valued functions square integrable on  $X$  with square integrable weak derivatives on  $X$ . It follows easily that  $H^{(k)}$  is self-adjoint for every  $k > 0$  (for example by the Kato-Rellich theorem, see [13], Theorem X.12). We define a unitary operator on  $\mathfrak{h}$  by

$$S = \frac{I - i\kappa_- E}{I + i\kappa_+ E}. \quad (8)$$

and an operator  $H$  on  $L^2(\mathbb{R}; \mathfrak{h})$  by

$$\begin{aligned} \text{Dom}(H) &= \{f \in W^{1,2}(\mathbb{R} \setminus \{0\}; \mathfrak{h}) : f(0^-) = Sf(0^+)\}, \\ Hf &= i \partial f. \end{aligned} \quad (9)$$

It follows easily that  $H$  is self-adjoint, compare [12], VIII.2, final example.

Remark: Any  $f \in W^{1,2}(\mathbb{R} \setminus \{0\}; \mathfrak{h})$  is absolutely continuous both on  $(-\infty, 0)$  and  $(0, \infty)$ , see for example [10], 2.6 Ex.6, but the exclusion of test functions supported at 0 allows jumps at 0. Higher dimensional situations ( $\mathbb{R}^n$  with  $n > 1$ ) are more complicated in this respect.

We define strongly continuous one-parameter groups of unitaries on  $L^2(\mathbb{R}; \mathfrak{h})$  by

$$U^{(k)}(t) = \exp(-itH^{(k)}), \quad U(t) = \exp(-itH).$$

We then have the following theorem.

**Theorem 5** *Let  $0 \leq T < \infty$ . Then*

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \left\| \left( U^{(k)}(t) - U(t) \right) f \right\| = 0, \quad \forall f \in L^2(\mathbb{R}; \mathfrak{h}).$$

We prove Theorem 5 at the end of this subsection. From the Trotter-Kato Theorem 3, it suffices to find, for every  $f \in \text{Dom}(H)$ , a sequence  $f^{(k)} \in \text{Dom}(H^{(k)})$  that satisfies condition (i) of Theorem 3.

If  $g$  is a  $\mathbb{C}$ -valued function on  $X$  and  $f \in L^2(X; \mathfrak{h}) \simeq \mathfrak{h} \otimes L^2(X; \mathbb{C})$  then we use the short notation  $gf$  for  $(I \otimes M_g)f$  where  $M_g$  is multiplication by  $g$ . With this convention we can also define  $g * f \in L^2(X; \mathfrak{h})$  and  $\langle g|f \rangle \in \mathfrak{h}$  for suitable functions  $g$ , using the same formulas as for  $\mathfrak{h} = \mathbb{C}$ .

**Definition 6** Let  $f$  be an element in the domain of  $H$ . Define an element  $f^{(k)}$  in the domain of  $H^{(k)}$  by

$$f^{(k)}(t) = (g^{(k)} * f)(t) = \int_{-\infty}^{\infty} g^{(k)}(t-s)f(s)ds.$$

**Lemma 7** Let  $\eta$  be an element of  $C(0, \infty)$  with compact support and let  $h$  be an element of  $W^{1,2}((0, \infty); \mathfrak{h}) \cap C^1((0, \infty); \mathfrak{h})$  such that  $h(0^+) = 0$ . Let  $\eta^{(k)}(x) = k\eta(kx)$  for all  $x \in (0, \infty)$  and  $k > 0$ . Then

$$\|\langle \eta^{(k)}|h \rangle\|_2 \leq \frac{C}{k}, \quad \forall k > 0,$$

for some positive constant  $C$ .

**Proof.** Note that the  $C^1$ -function  $h$  is Lipschitz on the support of  $\eta$ , that is, there exists a positive constant  $L$  such that

$$\|h(x) - h(y)\|_2 \leq L|x - y|, \quad \forall x, y \in \text{supp}(\eta),$$

where  $\text{supp}(\eta)$  denotes the support of  $\eta$ . Taking the limit for  $y$  to  $0^+$  gives

$$\|h(x)\|_2 \leq L|x|, \quad x \in \text{supp}(\eta).$$

We can define  $M := \max_{x \in (0, \infty)} |\eta(x)|$  and let  $N$  be a number to the right of the support of  $\eta$ . Now we have

$$\begin{aligned} \|\langle \eta^{(k)}|h \rangle\|_2 &\leq k \int_0^\infty |\eta(kx)| \|h(x)\|_2 dx \\ &\leq \frac{L}{k} \int_0^\infty |\eta(u)| u du \leq \frac{L}{k} \int_0^N M u du = \frac{LMN^2}{2k}. \end{aligned}$$

■

**Lemma 8** If  $f$  is in  $\text{Dom}(H) \cap C^\infty(\mathbb{R} \setminus \{0\}; \mathfrak{h})$ , and  $f^{(k)}$  is given by Definition 6, then we have

$$1. \quad \lim_{k \rightarrow \infty} \|f^{(k)} - f\|_2 = 0, \quad 2. \quad \lim_{k \rightarrow \infty} \|H^{(k)}f^{(k)} - Hf\|_2 = 0.$$

**Proof.** Note that the first limit follows immediately from a standard result on approximations by convolutions, see e.g. [11, Thm. 2.16]. For the second limit, note that

$$\partial(g^{(k)} * f) = g^{(k)} * \partial f + (f(0^+) - f(0^-))g^{(k)}, \quad (10)$$



Because  $\partial f = Hf$  and using [11, Thm. 2.16] once more, we find that

$$\lim_{k \rightarrow \infty} g^{(k)} * Hf = Hf.$$

That is, all we need to show is that

$$\lim_{k \rightarrow \infty} \left\| \left( if(0^+) - if(0^-) + \mathbb{E}\langle g^{(k)} | g^{(k)} * f \rangle \right) g^{(k)} \right\|_2 = 0. \quad (11)$$

Note that  $\langle g^{(k)} | g^{(k)} * f \rangle = \langle \rho^{(k)} | f \rangle$ . We can now apply Lemma 7 with  $h = f\chi_{(0,\infty)} - f(0^+)$  and  $\eta = \rho\chi_{(0,\infty)}$  (resp.  $h = f\chi_{(-\infty,0)} - f(0^-)$  and  $\eta = \rho\chi_{(-\infty,0)}$ ) to conclude that

$$\langle \rho^{(k)} | f \rangle \xrightarrow{k \rightarrow \infty} (\kappa_-)^* f(0^-) + (\kappa_+)^* f(0^+) = \kappa_+ f(0^-) + \kappa_- f(0^+),$$

with rate  $\frac{1}{k}$ . Using the boundary condition for  $f$ , we therefore find that

$$if(0^+) - if(0^-) + \mathbb{E}\langle g^{(k)} | g^{(k)} * f \rangle \longrightarrow i[(I - i\kappa_- \mathbb{E})f(0^+) - (I + i\kappa_+ \mathbb{E})f(0^-)] = 0,$$

with rate  $\frac{1}{k}$ . Note that the  $L^2$ -norm of  $g^{(k)}$  grows with rate  $\sqrt{k}$ , so that the limit in Eq. (11) follows. This completes the proof of the Lemma. ■

**Proof. [of Theorem 5]** The Theorem follows from a combination of the results in Theorem 3 and Lemma 8 and the fact that  $\text{Dom}(H) \cap C^\infty(\mathbb{R} \setminus \{0\}; \mathfrak{h})$  is a core for  $H$ . The latter follows from [11, Thm. 7.6]. ■

### 3 A Second Quantized Model

Let  $\mathbb{E}_{\alpha\beta}$  be bounded operators on  $\mathfrak{h}$  such that  $\mathbb{E}_{\alpha\beta}^\dagger = \mathbb{E}_{\beta\alpha}$  for  $\alpha, \beta \in \{0, 1\}$ . Consider the following family of operators on  $\mathfrak{h} \otimes \mathcal{F}$

$$H^{(k)} = id\Gamma(\partial) + \mathbb{E}_{11}A^\dagger(g^{(k)})A(g^{(k)}) + \mathbb{E}_{10}A^\dagger(g^{(k)}) + \mathbb{E}_{01}A(g^{(k)}) + \mathbb{E}_{00}, \quad (12)$$

choosing a suitable domain  $\text{Dom}(H^{(k)})$  of essential self-adjointness for all  $k > 0$ . (We conjecture that  $\mathfrak{h} \otimes \mathcal{E}(C_c^\infty(\mathbb{R}))$ , where  $\mathcal{E}(C_c^\infty(\mathbb{R}))$  is the set of exponential vectors  $e(f)$  with  $f \in C_c^\infty(\mathbb{R})$ , is a set of analytic vectors for the  $H^{(k)}$  but we haven't been able to prove this rigorously and leave it as an open problem.)

We denote the strongly continuous group of unitaries on  $\mathfrak{h} \otimes \mathcal{F}$  generated by the unique self-adjoint extension of  $H^{(k)}$  by  $U^{(k)}(t)$ . Let the triple  $(S, L, H)$  appearing in (4) be obtained from  $\mathbf{E} = (\mathbb{E}_{\alpha\beta})$  through (5): see (6).

The space  $\mathfrak{h} \otimes \mathcal{F} = \mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}))$  consists of vectors  $\Psi = (\Psi_m)_{m \geq 0}$  which are sequences of symmetric  $\mathfrak{h}$ -valued functions  $\Psi_m(t_1, \dots, t_m)$  where  $t_j \in \mathbb{R}$ . Following Gregoratti [4], we define the following spaces: (for  $I$  a Borel subset of  $\mathbb{R}$  and  $\mathfrak{H}$  a Hilbert space)

$$\begin{aligned}
\mathcal{H}^\Sigma(I^m, \mathfrak{H}) &= \left\{ v \in L^2(I^m, \mathfrak{H}) : \sum_{i=1}^m \partial_i v \in L^2(I^m, \mathfrak{H}) \right\}; \\
\mathcal{W} &= \left\{ \Psi \in \mathfrak{h} \otimes \mathcal{F} : \Psi_m \in \mathcal{H}^\Sigma(\mathbb{R}^m, \mathfrak{h}) : \sum_{m=0}^\infty \frac{1}{m!} \left\| \sum_{i=1}^m \partial_i \Psi_m \right\|^2 < \infty \right\}; \\
\mathcal{V}_s &= \left\{ \Psi \in \mathcal{W} : \sum_{m=0}^\infty \frac{1}{m!} \left\| \Psi_{m+1}(\cdot, t_{m+1} = s) \right\|^2 < \infty \right\}; \\
\mathcal{V}_{0\pm} &= \mathcal{V}_{0+} \cap \mathcal{V}_{0-}.
\end{aligned}$$

We remark that  $\mathcal{W}$  is the natural domain for  $d\Gamma(i\partial)$ . On  $\mathcal{V}_s$  we define the operators

$$(a(s)\Psi) = \Psi_{n+1}(\cdot, t_{n+1} = s).$$

On the subspace  $\mathcal{V}_{0\pm}$ , the operators  $d\Gamma(i\partial)$  and  $a(0^\pm)$  are all simultaneously defined.

**Definition 9 (The Gregoratti Hamiltonian)** *Define the following operator  $H$  on  $\mathfrak{h} \otimes \mathcal{F}$*

$$H\Phi = d\Gamma(i\partial_{ac})\Phi - i\mathbf{L}^\dagger \mathbf{S} a(0^+) \Phi + \left( \mathbf{H} - \frac{i}{2} \mathbf{L}^\dagger \mathbf{L} \right) \Phi, \quad (13)$$

$$\text{Dom}(H) = \{ \Phi \in \mathcal{V}_{0\pm} : a(0^-)\Phi = \mathbf{S} a(0^+)\Phi + \mathbf{L}\Phi \}. \quad (14)$$

It follows from the work of Chebotarev and Gregoratti [1, 4] that the operator  $H$  is essentially self-adjoint and its unique self-adjoint extension generates the unitary group  $U(t) = \Theta_t V_t$  where  $V_t$  is the unitary solution to the following quantum stochastic differential equation (3):

$$\begin{aligned}
dV(t) &= \left\{ (\mathbf{S} - 1)d\Lambda(t) + \mathbf{L}dA^\dagger(t) - \mathbf{L}^\dagger \mathbf{S} dA(t) - \frac{1}{2} \mathbf{L}^\dagger \mathbf{L} dt - i\mathbf{H}dt \right\} V(t), \\
V(0) &= I.
\end{aligned} \quad (15)$$

The main result of this section is the following theorem.

**Theorem 10** *Let  $0 \leq T < \infty$ . We have the following*

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \left\| \left( U^{(k)}(t) - U(t) \right) \Phi \right\| = 0, \quad \forall \Phi \in \mathfrak{h} \otimes \mathcal{F}.$$

Before proving the theorem (see the end of this section), we make some preparations. As in the previous section, we would like to use the Trotter-Kato Theorem, therefore, for every  $\Phi$  in a core for  $\text{Dom}(H)$ , we need to construct an approximating sequence  $\Phi^{(k)}$  that satisfies the first condition of Theorem 3. We again employ a smearing through convolution with  $g^{(k)}$ , this time applied as a second quantization.

**Definition 11** Let  $g^{(k)}$  be as in Definition 4 and assume further that  $g(t) \geq 0$  for all  $t$  (hence  $\|g\|_1 = 1$ ). Let  $G^{(k)} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  be the convolution with  $g^{(k)}$ , i.e.

$$G^{(k)}h = g^{(k)} * h, \quad \forall h \in L^2(\mathbb{R}).$$

Let  $\Phi$  be an element in  $\text{Dom}(H)$ . We define an element  $\Phi^{(k)}$  in the domain of  $H^{(k)}$  by

$$\Phi^{(k)} = \Gamma(G^{(k)})\Phi. \quad (16)$$

Here  $\Gamma(G^{(k)})$  denotes the second quantization of  $G^{(k)}$ .

Note that  $G^{(k)}$  is a contraction ( $\|g^{(k)}\|_1 = 1$ , i.e.  $\|\hat{g}^{(k)}\|_\infty \leq 1$  with  $\hat{g}^{(k)}$  the Fourier transform  $\hat{g}^{(k)} = \int_{-\infty}^{\infty} g^{(k)}(t)e^{-i\omega t}dt$ ), so its second quantization is well-defined). The positivity assumption on  $g$  implies that  $\kappa_+ = \kappa_- = \frac{1}{2}$  (which agrees with Section 1.2).

**Lemma 12** For all  $\Phi \in \mathfrak{h} \otimes \mathcal{F}$ , we have

$$\lim_{k \rightarrow \infty} \Gamma(G^{(k)})\Phi = \Phi.$$

**Proof.** Since the linear span of exponential vectors  $v \otimes e(h)$  is dense in  $\mathfrak{h} \otimes \mathcal{F}$  and  $\Gamma(G^{(k)})$  is bounded, it is enough to prove the Lemma for all vectors of the form  $\Phi = v \otimes e(h)$ . We have

$$\begin{aligned} & \|\Gamma(G^{(k)})v \otimes e(h) - v \otimes e(h)\|^2 = \\ & \|v\|^2 \left[ \exp(\|G^{(k)}h\|^2) + \exp(\|h\|^2) - \exp(\langle G^{(k)}h|h \rangle) - \exp(\langle h|G^{(k)}h \rangle) \right] \rightarrow 0, \end{aligned}$$

where in the last step we used [11, Thm. 2.16]. ■

We now recall the following result, see for instance [15].

**Lemma 13** Let  $C : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  be a contraction. We have for  $h \in L^2(\mathbb{R})$

$$\Gamma(C)\left(\text{Dom}(A(C^\dagger h))\right) \subset \text{Dom}(A(h)).$$

Moreover, on the domain of  $A(C^\dagger h)$ , we have

$$A(h)\Gamma(C) = \Gamma(C)A(C^\dagger h).$$

Note that we have the following second quantized version of equation (10):

$$d\Gamma(i\partial)\Phi^{(k)} = \Gamma(G^{(k)})d\Gamma(i\partial_{ac})\Phi + iA^\dagger(g^{(k)})\Gamma(G^{(k)})a_j\Phi,$$

where

$$(a_j\Phi)_m(t_1, \dots, t_m) = \Phi_{m+1}(t_1, \dots, t_m, 0^+) - \Phi_{m+1}(t_1, \dots, t_m, 0^-).$$

The action of  $H^{(k)}$  on  $\Phi^{(k)}$  can now be written as

$$\begin{aligned} H^{(k)}\Phi^{(k)} &= \Gamma(G^{(k)})d\Gamma(i\partial_{ac})\Phi \\ &\quad + A^\dagger(g^{(k)})\Gamma(G^{(k)})\left(ia_j\Phi + E_{11}A(\rho^{(k)})\Phi + E_{10}\Phi\right) \\ &\quad + E_{01}\Gamma(G^{(k)})A(\rho^{(k)})\Phi + E_{00}\Gamma(G^{(k)})\Phi. \end{aligned} \quad (17)$$

Here we have used Lemma 13 and the fact that  $A(G^{(k)\dagger}g^{(k)}) = A(\rho^{(k)})$ .

**Lemma 14** *The singular component of equation (17) converges strongly to zero as  $k \rightarrow \infty$ , i.e.,*

$$\left\| A^\dagger(g^{(k)})\Gamma(G^{(k)}) \left( ia_j\Phi + E_{11}A(\rho^{(k)})\Phi + E_{10}\Phi \right) \right\|_2 \xrightarrow{k \rightarrow \infty} 0,$$

for all  $\Phi$  in a core domain  $\mathcal{D}$  of  $H$ .

We defer the proof of this lemma to the next section.

Using Lemma 12, we find that the first term in Equation (17) converges to the first term in the Hamiltonian  $H$  given by Equation (13), i.e.

$$\lim_{k \rightarrow \infty} \left\| \Gamma(G^{(k)})d\Gamma(i\partial_{ac})\Phi - d\Gamma(i\partial_{ac})\Phi \right\|_2 = 0.$$

In the proof of the Lemma 14, it is shown that  $A(\rho^{(k)})\Phi$  converges in  $L^2$ -norm to  $\frac{1}{2}a(0^-)\Phi + \frac{1}{2}a(0^+)\Phi$ . Therefore, we find for the last line of Equation (17)

$$E_{01}\Gamma(G^{(k)})A(\rho^{(k)})\Phi + E_{00}\Gamma(G^{(k)})\Phi \longrightarrow E_{01} \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi + E_{00}\Phi.$$

Employing the boundary condition, we have that

$$\begin{aligned} & E_{01} \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi + E_{00}\Phi \\ &= E_{01} \left( \frac{1}{2}a(0^+) \Phi + \frac{1}{2} [S a(0^+) \Phi + L \Phi] \right) + E_{00}\Phi \\ &\equiv -iL^\dagger S a(0^+) \Phi + (H - \frac{i}{2}L^\dagger L)\Phi. \end{aligned}$$

Here we have used the algebraic identities

$$\begin{aligned} E_{01} \left( \frac{1}{2} + \frac{1}{2}S \right) &= E_{01} \left( \frac{1}{2} + \frac{1}{2} \frac{I - i\frac{1}{2}E_{11}}{I + i\frac{1}{2}E_{11}} \right) = E_{01} \frac{1}{I + i\frac{1}{2}E_{11}} \equiv -iL^\dagger S, \\ -i \frac{\frac{1}{2}}{I + i\frac{1}{2}E_{11}} &= \frac{1}{2} \text{Im} \left\{ \frac{\frac{1}{2}}{I + i\frac{1}{2}E_{11}} \right\} - \frac{i}{2} \frac{I}{I + i\frac{1}{2}E_{11}} \frac{I}{I - i\frac{1}{2}E_{11}}. \end{aligned}$$

Applying the Trotter-Kato Theorem, this completes the proof of our main result Theorem 10.

## 4 Proof of Lemma 14

Setting  $V^{(k)} = ia_j\Phi + E_{11}A(\rho^{(k)})\Phi + E_{10}\Phi$ , we see that

$$\begin{aligned} & \left\| A^\dagger(g^{(k)})\Gamma(G^{(k)})V^{(k)} \right\|_2^2 \\ &= \left\langle \Gamma(G^{(k)})V^{(k)} \left| A(g^{(k)})A^\dagger(g^{(k)})\Gamma(G^{(k)})V^{(k)} \right. \right\rangle \\ &= \left\langle \Gamma(G^{(k)})V^{(k)} \left| \left( A^\dagger(g^{(k)})A(g^{(k)}) + \|g^{(k)}\|_2^2 \right) \Gamma(G^{(k)})V^{(k)} \right. \right\rangle \\ &\leq \left\| A(g^{(k)})\Gamma(G^{(k)})V^{(k)} \right\|_2^2 + \|g^{(k)}\|_2^2 \|V^{(k)}\|_2^2, \end{aligned}$$

where in the last step we used that  $\Gamma(G^{(k)})$  is a contraction. We need to establish two further results: the first is that  $V^{(k)}$  goes to 0 sufficiently quickly and we prove this in Lemma 16 below; then we will have to show that this implies that the first term  $\|A(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2$  converges to 0 and we prove this in Lemma 17.

If we accept these results for the moment, then from the boundary conditions we have

$$\begin{aligned} & ia_j\Phi + E_{11} \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi + E_{10}\Phi \\ &= i \left( I - i\frac{1}{2}E_{11} \right) a(0^+) \Phi - i \left( I + i\frac{1}{2}E_{11} \right) a(0^-) \Phi + E_{10}\Phi \\ &= i \left( I + i\frac{1}{2}E_{11} \right) [Sa(0^+) \Phi + L\Phi - a(0^-) \Phi] = 0 \end{aligned}$$

so that, in fact,

$$V^{(k)} = E_{11} \left[ A(\rho^{(k)})\Phi - \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi \right]$$

As  $\|g^k\|_2$  grows at rate  $\sqrt{k}$ , it suffices to show that  $A(\rho^{(k)})\Phi - (\frac{1}{2}a(0^+) + \frac{1}{2}a(0^-))\Phi$  goes to 0 in norm with rate faster than  $\frac{1}{\sqrt{k}}$ . We will now establish this result below, but first we need to recall the definition of a pseudo-exponential vector from [4].

**Definition 15** Let  $F: t \mapsto F_t$  be a function from  $\mathbb{R}$  to  $\mathfrak{B}(\mathfrak{h})$  and define the corresponding pseudo-exponential vector  $\Psi(F, h)$  as

$$[\Psi(F, h)]_m(t_1, \dots, t_m) = \vec{T}F_{t_1} \cdots F_{t_m} h$$

for given  $h \in \mathfrak{h}$ , where  $\vec{T}$  denotes chronological ordering. That is

$$\vec{T}F_{t_1} \cdots F_{t_m} = F_{t_{\sigma(1)}} \cdots F_{t_{\sigma(m)}}$$

where  $\sigma$  is a permutation for which  $t_{\sigma(1)} \geq \cdots \geq t_{\sigma(m)}$ .

**Lemma 16** Let  $v \in W^{1,2}(\mathbb{R}/\{0\})$  and  $u \in W^{1,2}(\mathbb{R}/\{0\})$  with  $u|_{\mathbb{R}_+} = 0$  and  $u(0^-) = 1$ , then define  $F_t$  by

$$F_t = v(t) + u(t) [Sv(0^+) + L - v(0^-)] \quad (18)$$

then the domain  $\mathcal{D}$  of such pseudo-exponential vectors  $\Phi = \Psi(F, h)$  is a core for  $H$ . Moreover, for each such vector we have

$$\left\| A(\rho^{(k)})\Phi - \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi \right\|_2 = O\left(\frac{1}{k}\right).$$

**Proof.** The first part of this lemma is proved by Gregoratti where it is shown that  $\mathcal{D}$  is dense, and is contained in  $\text{Dom}(H) \cap \mathcal{V}_{0^\pm}$ , see [4] Propositions 4 and 5. Note that for  $\Phi = \Psi(F, h)$ , by (4) in [4] we have

$$\begin{aligned} a(t)\Phi &= v(t)\Phi, \quad t \in \{0^+\} \cup (0, \infty), \\ a(0^-)\Phi &= (Sv(0^+) + L)\Phi. \end{aligned}$$

To prove the second part, we begin by setting

$$\begin{aligned} Z_m(t_1, \dots, t_m) &= \left[ A(\rho^{(k)})\Phi - \left( \frac{1}{2}a(0^+) + \frac{1}{2}a(0^-) \right) \Phi \right]_m(t_1, \dots, t_m) \\ &= \int_0^\infty \rho^{(k)}(s) [\Phi_{m+1}(t_1, \dots, t_m, s) - \Phi_{m+1}(t_1, \dots, t_m, 0^+)] ds \\ &\quad + \int_{-\infty}^0 \rho^{(k)}(s) [\Phi_{m+1}(t_1, \dots, t_m, s) - \Phi_{m+1}(t_1, \dots, t_m, 0^-)] ds \\ &\equiv Z_m^+(t_1, \dots, t_m) + Z_m^-(t_1, \dots, t_m). \end{aligned}$$

We have  $\|Z_m\|^2 \leq (\|Z_m^+\| + \|Z_m^-\|)^2$  but

$$Z_m^+(t_1, \dots, t_m) = \int_0^\infty \rho^{(k)}(s) [v(s) - v(0^+)] ds \Phi_m(t_1, \dots, t_m)$$

and this prefactor is clearly  $O\left(\frac{1}{k}\right)$  from the argument used in Lemma 8.

However, we then have

$$\begin{aligned} &Z_m^-(t_1, \dots, t_m) \\ &= \int_{-\infty}^0 \rho^{(k)}(s) [F_{t_{\sigma(1)}} \cdots F_s \cdots F_{t_{\sigma(m)}} - F_{0^-} F_{t_{\sigma(1)}} \cdots F_{t_{\sigma(m)}}] h ds \end{aligned}$$

where  $\sigma$  is the chronological time ordering permutation.

We note however that  $[F_t, F_s] = 0$  for all  $t, s$ , therefore we have

$$\begin{aligned} Z_m^-(t_1, \dots, t_m) &= \int_{-\infty}^0 \rho^{(k)}(s) [F_s - F_{0^-}] F_{t_{\sigma(1)}} \cdots F_{t_{\sigma(m)}} h ds \\ &= \int_{-\infty}^0 \rho^{(k)}(s) [u(s) - u(0^-)] [Sv(0^+) + L - v(0^-)] \\ &\quad \times F_{t_{\sigma(1)}} \cdots F_{t_{\sigma(m)}} h ds \end{aligned}$$

where we used (18). From the argument in Lemma 8 again, we see that this is  $O\left(\frac{1}{k}\right)$ . ■

**Lemma 17** *For  $\Phi$  chosen as a pseudo-exponential vector, as in Lemma 16, we have that  $\|A(g^{(k)})\Gamma(G^{(k)})V^{(k)}\|_2^2$  converges to 0 as  $k \rightarrow \infty$ .*

**Proof.** We have that

$$A(g^{(k)})\Gamma(G^{(k)})V^{(k)} = \Gamma(G^{(k)})A(\rho^{(k)})V^{(k)},$$

with  $\Gamma(G^{(k)})$  a contraction. The  $m$ th level of the Fock space component of  $A(\rho^{(k)})V^{(k)}$  may be written as

$$E_{11}A(\rho^{(k)})Z_m^+ + E_{11}A(\rho^{(k)})Z_m^-,$$

where we use the same conventions as in Lemma 16. The first term has the explicit components

$$\begin{aligned} & E_{11} \int dt \rho^{(k)}(t) \int_0^\infty \rho^{(k)}(s) [v(s) - v(0^+)] ds \Phi_{m+1}(t, t_1, \dots, t_m) \\ &= E_{11} \int dt \rho^{(k)}(t) F_t \int_0^\infty \rho^{(k)}(s) [v(s) - v(0^+)] ds \Phi_m(t_1, \dots, t_m) \end{aligned}$$

which is norm bounded by  $\|E_{11}\| \int dt \rho^{(k)}(t) \|F_t\| \|Z_m^+\|$ , and we note that in fact  $\int dt \rho^{(k)}(t) \|F_t\| = \int d\tau \rho(\tau) \|F_{\tau/k}\|$ . An equivalent bound is easily shown to hold for  $E_{11}A(\rho^{(k)})Z_m^-$  and so by an argument similar to lemma 16 we obtain the desired result. ■

## Epilogue

After completion of this work, the authors became aware of the book by W. von Waldenfels [16] which gives a complete resolvent analysis of the Chebotarev-Gregoratti-von Waldenfels Hamiltonian, and in the final chapter describes a strong resolvent limit by colored noise approximations. The convergence is comparable to the strong uniform convergence considered here, but the approach is very different.

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